2. Examples

Linear programming is a wonderful tool. But in order to use it, one first has to start suspecting that the considered computational problem might be expressible by a linear program, and then one has to really express it that way. In other words, one has to see linear programming "behind the scenes."

One of the main goals of this book is to help the reader acquire skills in this direction. We believe that this is best done by studying diverse examples and by practice. In this chapter we present several basic cases from the wide spectrum of problems amenable to linear programming methods, and we demonstrate a few tricks for reformulating problems that do not look like linear programs at first sight. Further examples are covered in Chapter 3, and Chapter 8 includes more advanced applications.

Once we have a suitable linear programming formulation (a "model" in the mathematical programming parlance), we can employ general algorithms. From a programmer's point of view this is very convenient, since it suffices to input the appropriate objective function and constraints into general-purpose software.

If efficiency is a concern, this need not be the end of the story. Many problems have special features, and sometimes specialized algorithms are known, or can be constructed, that solve such problems substantially faster than a general approach based on linear programming. For example, the study of network flows, which we consider in Section 2.2, constitutes an extensive subfield of theoretical computer science, and fairly efficient algorithms have been developed. Computing a maximum flow via linear programming is thus not the best approach for large-scale instances.

However, even for problems where linear programming doesn't ultimately yield the most efficient available algorithm, starting with a linear programming formulation makes sense: for fast prototyping, case studies, and deciding whether developing problem-specific software is worth the effort.

2.1 Optimized Diet: Wholesome and Cheap?

. . . and when Rabbit said, "Honey or condensed milk with your bread?" he was so excited that he said, "Both," and then, so as not to seem greedy, he added, "But don't bother about the bread, please."

A. A. Milne, Winnie the Pooh

The Office of Nutrition Inspection of the EU recently found out that dishes served at the dining and beverage facility "Bullneck's," such as herring, hot dogs, and house-style hamburgers do not comport with the new nutritional regulations, and its report mentioned explicitly the lack of vitamins A and C and dietary fiber. The owner and operator of the aforementioned facility is attempting to rectify these shortcomings by augmenting the menu with vegetable side dishes, which he intends to create from white cabbage, carrots, and a stockpile of pickled cucumbers discovered in the cellar. The following table summarizes the numerical data: the prescribed amount of the vitamins and fiber per dish, their content in the foods, and the unit prices of the foods.[1]

Food	Carrot, Raw	White Cabbage, Raw	Cucumber, Pickled	Required per dish
Vitamin A [mg/kg]	35	0.5	0.5	0.5 mg
Vitamin C [mg/kg]	60	300	10	15 mg
Dietary Fiber [g/kg]	30	20	10	4 g
price [€/kg]	0.75	0.5	0.15*	—

*Residual accounting price of the inventory, most likely unsaleable.

At what minimum additional price per dish can the requirements of the Office of Nutrition Inspection be satisfied? This question can be expressed by the following linear program:

$$\begin{array}{ll} \text{Minimize} & 0.75x_1 + 0.5x_2 + 0.15x_3 \\ \text{subject to} & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_3 \geq 0 \\ & 35x_1 + 0.5x_2 + 0.5x_3 \geq 0.5 \\ & 60x_1 + 300x_2 + 10x_3 \geq 15 \\ & 30x_1 + 20x_2 + 10x_3 \geq 4. \end{array}$$

The variable x_1 specifies the amount of carrot (in kg) to be added to each dish, and similarly for x_2 (cabbage) and x_3 (cucumber). The objective function

[1] For those interested in healthy diet: The vitamin contents and other data are more or less realistic.

expresses the price of the combination. The amounts of carrot, cabbage, and cucumber are always nonnegative, which is captured by the conditions $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ (if we didn't include them, an optimal solution might perhaps have the amount of carrot, say, negative, by which one would seemingly save money). Finally, the inequalities in the last three lines force the requirements on vitamins A and C and of dietary fiber.

The linear program can be solved by standard methods. The optimal solution yields the price of €0.07 with the following doses: carrot 9.5 g, cabbage 38 g, and pickled cucumber 290 g per dish (all rounded to two significant digits). This probably wouldn't pass another round of inspection. In reality one would have to add further constraints, for example, one on the maximum amount of pickled cucumber.

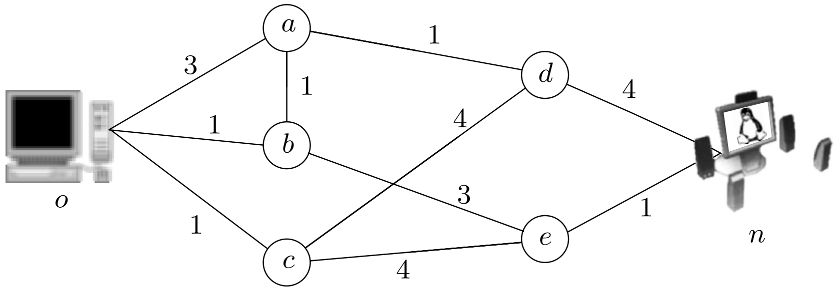
We have included this example so that our treatment doesn't look too revolutionary. It seems that all introductions to linear programming begin with various dietary problems, most likely because the first large-scale problem on which the simplex method was tested in 1947 was the determination of an adequate diet of least cost. Which foods should be combined and in what amounts so that the required amounts of all essential nutrients are satisfied and the daily ration is the cheapest possible. The linear program had 77 variables and 9 constraints, and its solution by the simplex method using hand-operated desk calculators took approximately 120 man-days.

Later on, when George Dantzig had already gained access to an electronic computer, he tried to optimize his own diet as well. The optimal solution of the first linear program that he constructed recommended daily consumption of several liters of vinegar. When he removed vinegar from the next input, he obtained approximately 200 bouillon cubes as the basis of the daily diet. This story, whose truth is not entirely out of the question, doesn't diminish the power of linear programming in any way, but it illustrates how difficult it is to capture mathematically all the important aspects of real-life problems. In the realm of nutrition, for example, it is not clear even today what exactly the influence of various components of food is on the human body. (Although, of course, many things *are* clear, and hopes that the science of the future will recommend hamburgers as the main ingredient of a healthy diet will almost surely be disappointed.) Even if it were known perfectly, few people want and can formulate exactly what they expect from their diet—apparently, it is much easier to formulate such requirements for the diet of someone else. Moreover, there are nonlinear dependencies among the effects of various nutrients, and so the dietary problem can never be captured perfectly by linear programming.

There are many applications of linear programming in industry, agriculture, services, etc. that from an abstract point of view are variations of the diet problem and do not introduce substantially new mathematical tricks. It may still be challenging to design good models for real-life problems of this kind, but the challenges are not mathematical. We will not dwell on such problems here (many examples can be found in Chvátal's book cited in Chapter 9), and we will present problems in which the use of linear programming has different flavors.

2.2 Flow in a Network

An administrator of a computer network convinced his employer to purchase a new computer with an improved sound system. He wants to transfer his music collection from an old computer to the new one, using a local network. The network looks like this:



What is the maximum transfer rate from computer o (old) to computer n (new)? The numbers near each data link specify the maximum transfer rate of that link (in Mbit/s, say). We assume that each link can transfer data in either direction, but not in both directions simultaneously. So, for example, through the link ab one can *either* send data from a to b at any rate from 0 up to 1 Mbit/s, *or* send data from b to a at any rate from 0 to 1 Mbit/s.

The nodes $a, b, \ldots, e$ are not suitable for storing substantial amounts of data, and hence all data entering them has to be sent further immediately. From this we can already see that the maximum transfer rate cannot be used on all links simultaneously (consider node a, for example). Thus we have to find an appropriate value of the data flow for each link so that the total transfer rate from o to n is maximum.

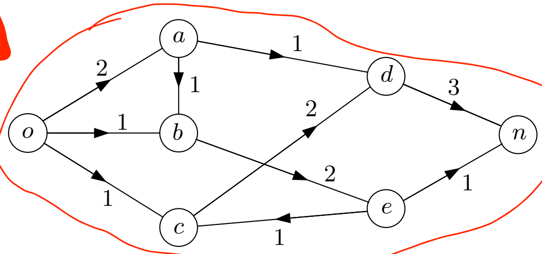
For every link in the network we introduce one variable. For example, x_{be} specifies the rate by which data is transfered from b to e. Here x_{be} can also be negative, which means that data flow in the opposite direction, from e to b. (And we thus do *not* introduce another variable x_{eb}, which would correspond to the transfer rate from e to b.) There are 10 variables: x_{oa}, x_{ob}, x_{oc}, x_{ab}, x_{ad}, x_{be}, x_{cd}, x_{ce}, x_{dn}, and x_{en}.

We set up the following linear program:

$$
\begin{array}{ll}
\text{Maximize} & x_{oa} + x_{ob} + x_{oc} \\
\text{subject to} & -3 \le x_{oa} \le 3, \quad -1 \le x_{ob} \le 1, \quad -1 \le x_{oc} \le 1 \\
& -1 \le x_{ab} \le 1, \quad -1 \le x_{ad} \le 1, \quad -3 \le x_{be} \le 3 \\
& -4 \le x_{cd} \le 4, \quad -4 \le x_{ce} \le 4, \quad -4 \le x_{dn} \le 4 \\
& -1 \le x_{en} \le 1 \\
& \qquad x_{oa} = x_{ab} + x_{ad} \\
& x_{ob} + x_{ab} = x_{be} \\
& \qquad x_{oc} = x_{cd} + x_{ce} \\
& x_{ad} + x_{cd} = x_{dn} \\
& x_{be} + x_{ce} = x_{en}.
\end{array}
$$

The objective function $x_{oa} + x_{ob} + x_{oc}$ expresses the total rate by which data is sent out from computer o. Since it is neither stored nor lost (hopefully) anywhere, it has to be received at n at the same rate. The next 10 constraints, $-3 \le x_{oa} \le 3$ through $-1 \le x_{en} \le 1$, restrict the transfer rates along the individual links. The remaining constraints say that whatever enters each of the nodes a through e has to leave immediately.

The optimal solution of this linear program is depicted below:



The number near each link is the transfer rate on that link, and the arrow determines the direction of the data flow. Note that between c and e data has to be sent in the direction from e to c, and hence $x_{ce} = -1$. The optimum value of the objective function is 4, and this is the desired maximum transfer rate.

In this example it is easy to see that the transfer rate cannot be larger, since the total capacity of all links connecting the computers o and a to the rest of the network equals 4. This is a special case of a remarkable theorem on maximum flow and minimum cut, which is usually discussed in courses on graph algorithms (see also Section 8.2).

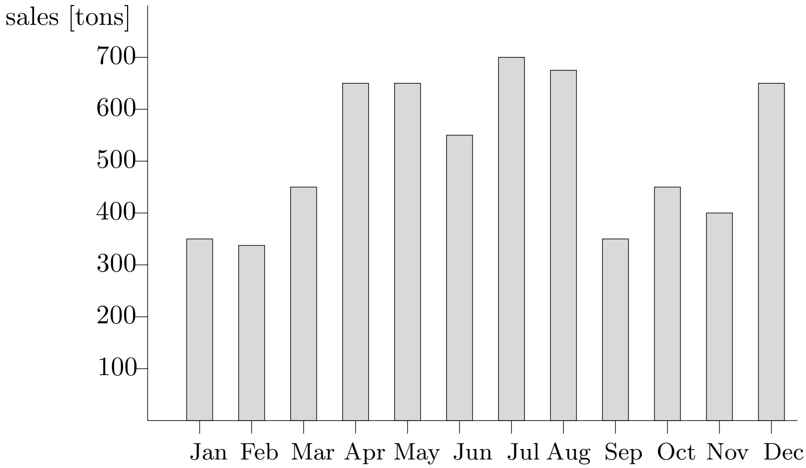
Our example of data flow in a network is small and simple. In practice, however, flows are considered in intricate networks, sometimes even with many source nodes and sink nodes. These can be electrical networks (current flows), road or railroad networks (cars or trains flow), telephone networks (voice or data signals flow), financial (money flows), and so on. There are also many less-obvious applications of network flows—for example, in image processing.

Historically, the network flow problem was first formulated by American military experts in search of efficient ways of disrupting the railway system of the Soviet block; see

> A. Schrijver: On the history of the transportation and maximum flow problems, *Math. Programming Ser. B* 91(2002) 437–445.

2.3 Ice Cream All Year Round

The next application of linear programming again concerns food (which should not be surprising, given the importance of food in life and the difficulties in optimizing sleep or love). The ice cream manufacturer Icicle Works Ltd.[2] needs to set up a production plan for the next year. Based on history, extensive surveys, and bird observations, the marketing department has come up with the following prediction of monthly sales of ice cream in the next year:



Now Icicle Works Ltd. needs to set up a production schedule to meet these demands.

A simple solution would be to produce "just in time," meaning that all the ice cream needed in month i is also produced in month i, $i = 1, 2, \ldots, 12$. However, this means that the produced amount would vary greatly from month to month, and a change in the produced amount has significant costs: Temporary workers have to be hired or laid off, machines have to be adjusted,

[2] Not to be confused with a rock group of the same name. The name comes from a nice science fiction story by Frederik Pohl.

and so on. So it would be better to spread the production more evenly over the year: In months with low demand, the idle capacities of the factory could be used to build up a stock of ice cream for the months with high demand.

So another simple solution might be a completely “flat” production schedule, with the same amount produced every month. Some thought reveals that such a schedule need not be feasible if we want to end up with zero surplus at the end of the year. But even if it is feasible, it need not be ideal either, since storing ice cream incurs a nontrivial cost. It seems likely that the best production schedule should be somewhere between these two extremes (production following demand and constant production). We want a compromise minimizing the total cost resulting both from changes in production and from storage of surpluses.

To formalize this problem, let us denote the demand in month i by $d_i \geq 0$ (in tons). Then we introduce a nonnegative variable x_i for the production in month i and another nonnegative variable s_i for the total surplus in store at the end of month i. To meet the demand in month i, we may use the production in month i and the surplus at the end of month $i-1$:

$$x_i + s_{i-1} \geq d_i \quad \text{for } i = 1, 2, \ldots, 12.$$

The quantity $x_i + s_{i-1} - d_i$ is exactly the surplus after month i, and thus we have

$$x_i + s_{i-1} - s_i = d_i \quad \text{for } i = 1, 2, \ldots, 12.$$

Assuming that initially there is no surplus, we set $s_0 = 0$ (if we took the production history into account, s_0 would be the surplus at the end of the previous year). We also set $s_{12} = 0$, unless we want to plan for another year.

Among all nonnegative solutions to these equations, we are looking for one that minimizes the total cost. Let us assume that changing the production by 1 ton from month $i-1$ to month i costs €50, and that storage facilities for 1 ton of ice cream cost €20 per month. Then the total cost is expressed by the function

$$50 \sum_{i=1}^{12} |x_i - x_{i-1}| + 20 \sum_{i=1}^{12} s_i,$$

where we set $x_0 = 0$ (again, history can easily be taken into account).

Unfortunately, this cost function is not linear. Fortunately, there is a simple but important trick that allows us to make it linear, at the price of introducing extra variables.

The change in production is either an increase or a decrease. Let us introduce a nonnegative variable y_i for the increase from month $i-1$ to month i, and a nonnegative variable z_i for the decrease. Then

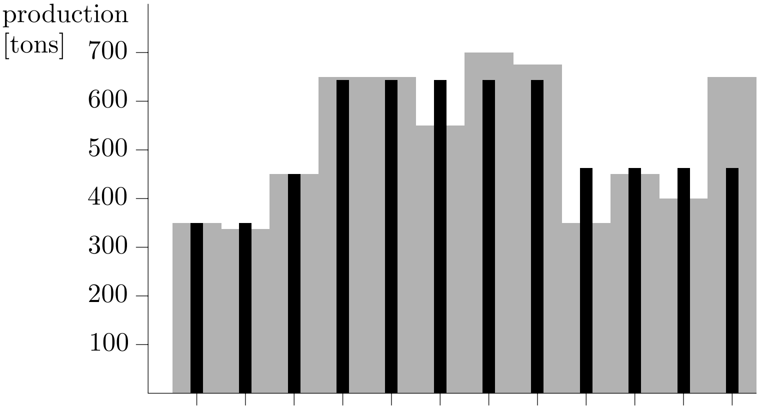
$$x_i - x_{i-1} = y_i - z_i \text{ and } |x_i - x_{i-1}| = y_i + z_i.$$

A production schedule of minimum total cost is given by an optimal solution of the following linear program:

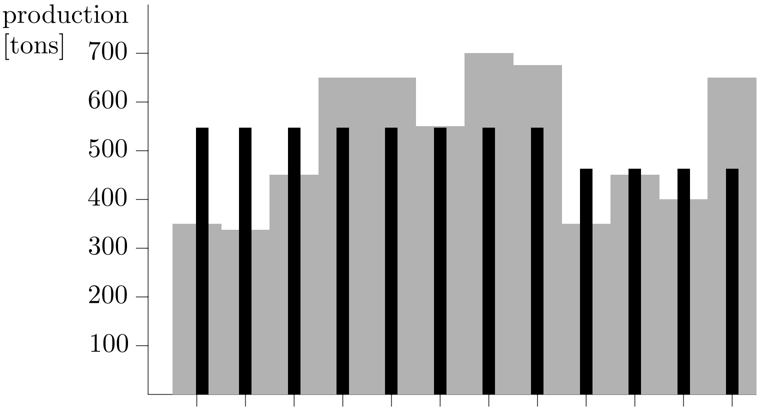
$$\begin{array}{ll} \text{Minimize} & 50\sum_{i=1}^{12} y_i + 50\sum_{i=1}^{12} z_i + 20\sum_{i=1}^{12} s_i \\ \text{subject to} & x_i + s_{i-1} - s_i = d_i \text{ for } i = 1, 2, \ldots, 12 \\ & x_i - x_{i-1} = y_i - z_i \text{ for } i = 1, 2, \ldots, 12 \\ & x_0 = 0 \\ & s_0 = 0 \\ & s_{12} = 0 \\ & x_i, s_i, y_i, z_i \geq 0 \text{ for } i = 1, 2, \ldots, 12. \end{array}$$

To see that an optimal solution $(\mathbf{s}^*, \mathbf{y}^*, \mathbf{z}^*)$ of this linear program indeed defines a schedule, we need to note that one of y_i^* and z_i^* has to be zero for all i, for otherwise, we could decrease both and obtain a better solution. This means that $y_i^* + z_i^*$ indeed equals the change in production from month $i-1$ to month i, as required.

In the Icicle Works example above, this linear program yields the following production schedule (shown with black bars; the gray background graph represents the demands).



Below is the schedule we would get with zero storage costs (that is, after replacing the "20" by "0" in the above linear program).

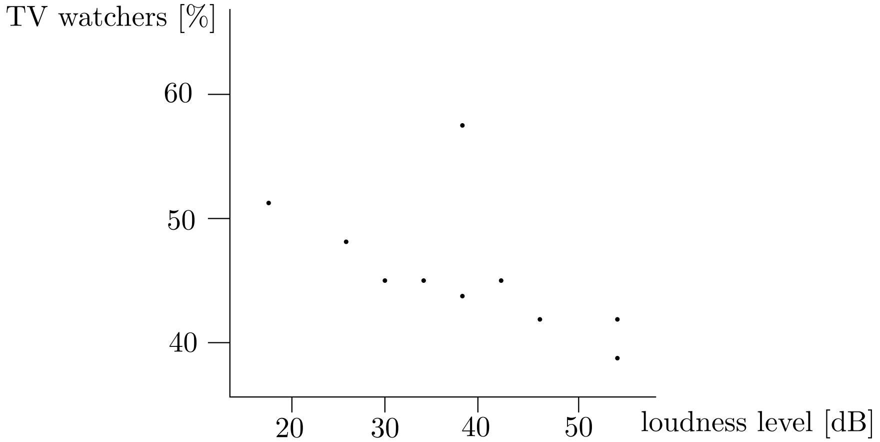


The pattern of this example is quite general, and many problems of optimal control can be solved via linear programming in a similar manner. A neat example is "Moon Rocket Landing," a once-popular game for programmable calculators (probably not sophisticated enough to survive in today's competition). A lunar module with limited fuel supply is descending vertically to the lunar surface under the influence of gravitation, and at chosen time intervals it can flash its rockets to slow down the descent (or even to start flying upward). The goal is to land on the surface with (almost) zero speed before exhausting all of the fuel. The reader is invited to formulate an appropriate linear program for determining the minimum amount of fuel necessary for landing, given the appropriate input data. For the linear programming formulation, we have to discretize time first (in the game this was done anyway), but with short enough time steps this doesn't make a difference in practice.

Let us remark that this particular problem can be solved analytically, with some calculus (or even mathematical control theory). But in even slightly more complicated situations, an analytic solution is out of reach.

2.4 Fitting a Line

The loudness level of nightingale singing was measured every evening for a number of days in a row, and the percentage of people watching the principal TV news was surveyed by questionnaires. The following diagram plots the measured values by points in the plane:



The simplest dependencies are linear, and many dependencies can be well approximated by a linear function. We thus want to find a line that best fits the measured points. (Readers feeling that this example is not sufficiently realistic can recall some measurements in physics labs, where the measured quantities should actually obey an exact linear dependence.)

How can one formulate mathematically that a given line "best fits" the points? There is no unique way, and several different criteria are commonly used for line fitting in practice.

The most popular one is the method of *least squares*, which for given points $(x_1, y_1),\ldots, (x_n, y_n)$ seeks a line with equation $y = ax + b$ minimizing the expression

$$\sum_{i=1}^{n}(ax_i + b - y_i)^2. \tag{2.1}$$

In words, for every point we take its vertical distance from the line, square it, and sum these "squares of errors."

This method need not always be the most suitable. For instance, if a few exceptional points are measured with very large error, they can influence the resulting line a great deal. An alternative method, less sensitive to a small number of "outliers," is to minimize the sum of absolute values of all errors:

$$\sum_{i=1}^{n}|ax_i + b - y_i|. \tag{2.2}$$

By a trick similar to the one we have seen in Section 2.3, this apparently nonlinear optimization problem can be captured by a linear program:

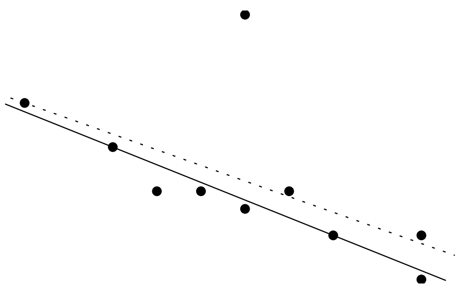
$$\begin{array}{lll} \text{Minimize} & e_1 + e_2 + \cdots + e_n & \\ \text{subject to} & e_i \geq ax_i + b - y_i & \text{for } i = 1, 2, \ldots, n \\ & e_i \geq -(ax_i + b - y_i) & \text{for } i = 1, 2, \ldots, n. \end{array}$$

The variables are a, b, and $e_1, e_2, \ldots, e_n$ (while $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are given numbers). Each e_i is an auxiliary variable standing for the error at the ith point. The constraints guarantee that

$$e_i \geq \max\Big(ax_i + b - y_i, -(ax_i + b - y_i)\Big) = |ax_i + b - y_i|.$$

In an optimal solution each of these inequalities has to be satisfied with equality, for otherwise, we could decrease the corresponding e_i. Thus, an optimal solution yields a line minimizing the expression (2.2).

The following picture shows a line fitted by this method (solid) and a line fitted using least squares (dotted):



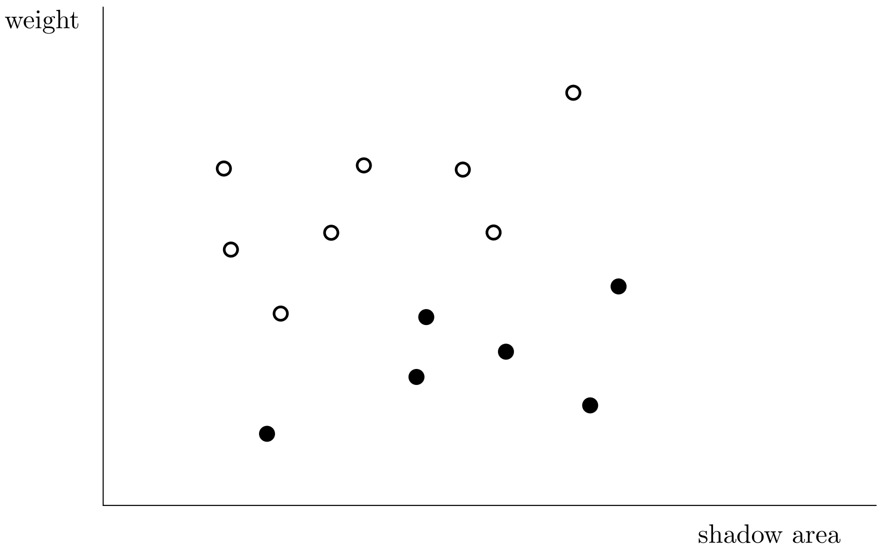
In conclusion, let us recall the useful trick we have learned here and in the previous section:

Objective functions or constraints involving absolute values can often be handled via linear programming by introducing extra variables or extra constraints.

2.5 Separation of Points

A computer-controlled rabbit trap "Gromit RT 2.1" should be programmed so that it catches rabbits, but if a weasel wanders in, it is released. The trap can weigh the animal inside and also can determine the area of its shadow.

These two parameters were collected for a number of specimens of rabbits and weasels, as depicted in the following graph:



(empty circles represent rabbits and full circles weasels).

Apparently, neither weight alone nor shadow area alone can be used to tell a rabbit from a weasel. One of the next-simplest things would be a linear criterion distinguishing them. That is, geometrically, we would like to separate the black points from the white points by a straight line if possible. Mathematically speaking, we are given m white points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$

and n black points $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ in the plane, and we would like to find out whether there exists a line having all white points on one side and all black points on the other side (none of the points should lie on the line).

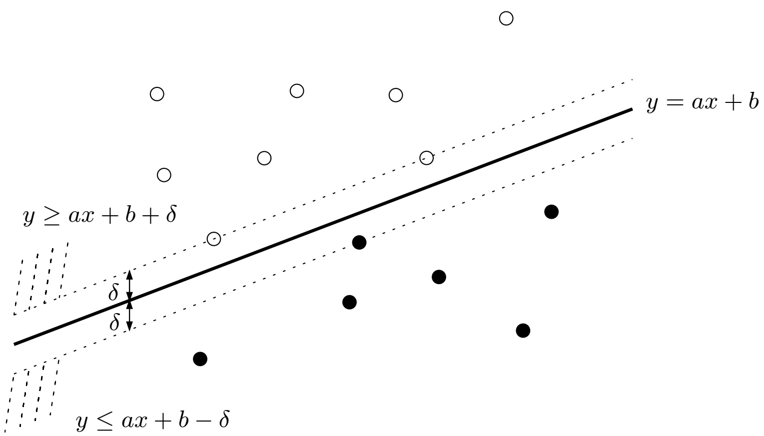
In a solution of this problem by linear programming we distinguish three cases. First we test whether there exists a *vertical* line with the required property. This case needs neither linear programming nor particular cleverness.

The next case is the existence of a line that is not vertical and that has all black points below it and all white points above it. Let us write the equation of such a line as $y = ax + b$, where a and b are some yet unknown real numbers. A point $\mathbf{r}$ with coordinates $x(\mathbf{r})$ and $y(\mathbf{r})$ lies above this line if $y(\mathbf{r}) > ax(\mathbf{r}) + b$, and it lies below it if $y(\mathbf{r}) < ax(\mathbf{r}) + b$. So a suitable line exists if and only if the following system of inequalities with variables a and b has a solution:

$$\begin{aligned} y(\mathbf{p}_i) &> ax(\mathbf{p}_i) + b \qquad \text{for } i = 1, 2, \ldots, m \\ y(\mathbf{q}_j) &< ax(\mathbf{q}_j) + b \qquad \text{for } j = 1, 2, \ldots, n. \end{aligned}$$

We haven't yet mentioned strict inequalities in connection with linear programming, and actually, they are not allowed in linear programs. But here we can get around this issue by a small trick: We introduce a new variable δ, which stands for the "gap" between the left and right sides of each strict inequality. Then we try to make the gap as large as possible:

$$\begin{array}{lll} \text{Maximize} & \delta & \\ \text{subject to} & y(\mathbf{p}_i) \geq ax(\mathbf{p}_i) + b + \delta & \text{for } i = 1, 2, \ldots, m \\ & y(\mathbf{q}_j) \leq ax(\mathbf{q}_j) + b - \delta & \text{for } j = 1, 2, \ldots, n. \end{array}$$



This linear program has three variables: a, b, and δ. The optimal δ is positive exactly if the preceding system of strict inequalities has a solution, and the latter happens exactly if a nonvertical line exists with all black points below and all white points above.

Similarly, we can deal with the third case, namely the existence of a nonvertical line having all black points above it and all white points below it. This completes the description of an algorithm for the line separation problem.

A plane separating two point sets in $\mathbb{R}^3$ can be computed by the same approach, and we can also solve the analogous problem in higher dimensions. So we could try to distinguish rabbits from weasels based on more than two measured parameters.

Here is another, perhaps more surprising, extension. Let us imagine that separating rabbits from weasels by a straight line proved impossible. Then we could try, for instance, separating them by a graph of a quadratic function (a parabola), of the form $ax^2 + bx + c$. So given m white points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$ and n black points $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ in the plane, we now ask, are there coefficients $a, b, c \in \mathbb{R}$ such that the graph of $f(x) = ax^2 + bx + c$ has all white points above it and all black points below? This leads to the inequality system

$$\begin{aligned} y(\mathbf{p}_i) &> ax(\mathbf{p}_i)^2 + bx(\mathbf{p}_i) + c \qquad \text{for } i = 1, 2, \ldots, m \\ y(\mathbf{q}_j) &< ax(\mathbf{q}_j)^2 + bx(\mathbf{q}_j) + c \qquad \text{for } j = 1, 2, \ldots, n. \end{aligned}$$

By introducing a gap variable δ as before, this can be written as the following linear program in the variables a, b, c, and δ:

$$\begin{array}{lll} \text{Maximize} & \delta & \\ \text{subject to} & y(\mathbf{p}_i) \geq ax(\mathbf{p}_i)^2 + bx(\mathbf{p}_i) + c + \delta & \text{for } i = 1, 2, \ldots, m \\ & y(\mathbf{q}_j) \leq ax(\mathbf{q}_j)^2 + bx(\mathbf{q}_j) + c - \delta & \text{for } j = 1, 2, \ldots, n. \end{array}$$

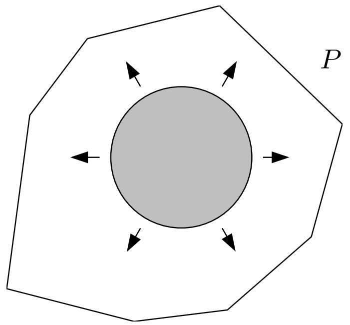
In this linear program the quadratic terms are coefficients and therefore they cause no harm.

The same approach also allows us to test whether two point sets in the plane, or in higher dimensions, can be separated by a function of the form $f(\mathbf{x}) = a_1\varphi_1(\mathbf{x}) + a_2\varphi_2(\mathbf{x}) + \cdots + a_k\varphi_k(\mathbf{x})$, where $\varphi_1, \ldots, \varphi_k$ are given functions (possibly nonlinear) and $a_1, a_2, \ldots, a_k$ are real coefficients, in the sense that $f(\mathbf{p}_i) > 0$ for every white point $\mathbf{p}_i$ and $f(\mathbf{q}_j) < 0$ for every black point $\mathbf{q}_j$.

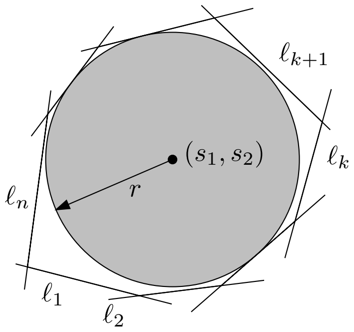
2.6 Largest Disk in a Convex Polygon

Here we will encounter another problem that may look nonlinear at first sight but can be transformed to a linear program. It is a simple instance of a geometric *packing problem:* Given a container, in our case a convex polygon, we want to fit as large an object as possible into it, in our case a disk of the largest possible radius.

Let us call the given convex polygon P, and let us assume that it has n sides. As we said, we want to find the largest circular disk contained in P.



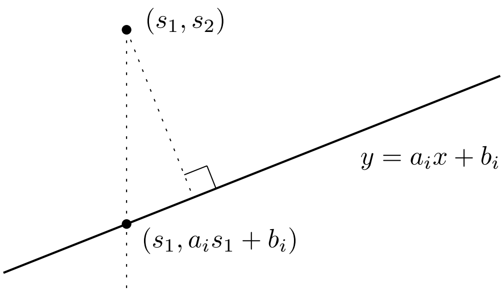
For simplicity let us assume that none of the sides of P is vertical. Let the ith side of P lie on a line ℓ_i with equation $y = a_i x + b_i$, $i = 1, 2, \ldots, n$, and let us choose the numbering of the sides in such a way that the first, second, up to the kth side bound P from below, while the $(k+1)$st through nth side bound it from above.



Let us now ask, under what conditions does a circle with center $\mathbf{s} = (s_1, s_2)$ and radius r lie completely inside P? This is the case if and only if the point $\mathbf{s}$ has distance at least r from each of the lines $\ell_1, \ldots, \ell_n$, lies above the lines $\ell_1, \ldots, \ell_k$, and lies below the lines $\ell_{k+1}, \ldots, \ell_n$. We compute the distance of $\mathbf{s}$ from ℓ_i. A simple calculation using similarity of triangles and the Pythagorean theorem shows that this distance equals the absolute value of the expression

$$\frac{s_2 - a_i s_1 - b_i}{\sqrt{a_i^2 + 1}}.$$

Moreover, the expression is positive if $\mathbf{s}$ lies above ℓ_i, and it is negative if $\mathbf{s}$ lies below ℓ_i:



The disk of radius r centered at $\mathbf{s}$ thus lies inside P exactly if the following system of inequalities is satisfied:

$$\begin{aligned} \frac{s_2 - a_i s_1 - b_i}{\sqrt{a_i^2 + 1}} &\geq r, \qquad i = 1, 2, \ldots, k \\ \frac{s_2 - a_i s_1 - b_i}{\sqrt{a_i^2 + 1}} &\leq -r, \quad i = k+1, k+2, \ldots, n. \end{aligned}$$

Therefore, we want to find the largest r such that there exist s_1 and s_2 so that all the constraints are satisfied. This yields a linear program! (Some might be frightened by the square roots, but these can be computed in advance, since all the a_i are concrete numbers.)

$$\begin{array}{lrcll} \text{Maximize} & r & & & \\ \text{subject to} & \dfrac{s_2 - a_i s_1 - b_i}{\sqrt{a_i^2 + 1}} & \geq & r & \text{for } i = 1, 2, \ldots, k \\ & \dfrac{s_2 - a_i s_1 - b_i}{\sqrt{a_i^2 + 1}} & \leq & -r & \text{for } i = k+1, k+2, \ldots, n. \end{array}$$

There are three variables: s_1, s_2, and r. An optimal solution yields the desired largest disk contained in P.

A similar problem in higher dimension can be solved analogously. For example, in three-dimensional space we can ask for the largest ball that can be placed into the intersection of n given half-spaces.

Interestingly, another similar-looking problem, namely, finding the smallest disk containing a given convex n-gon in the plane, cannot be expressed by a linear program and has to be solved differently; see Section 8.7.

Both in practice and in theory, one usually encounters geometric packing problems that are more complicated than the one considered in this section and not so easily solved by linear programming. Often we have a fixed collection of objects and we want to pack as many of them as possible into a given container (or several containers). Such problems are encountered by confectioners when cutting cookies from a piece of dough, by tailors or clothing manufacturers when making as many trousers, say, as possible from a large piece of cloth, and so on. Typically, these problems are computationally hard, but linear programming can sometimes help in devising heuristics or approximate algorithms.

2.7 Cutting Paper Rolls

Here we have another industrial problem, and the application of linear programming is quite nonobvious. Moreover, we will naturally encounter an integrality constraint, which will bring us to the topic of the next chapter.

A paper mill manufactures rolls of paper of a standard width 3 meters. But customers want to buy paper rolls of shorter width, and the mill has to cut such rolls from the 3 m rolls. One 3 m roll can be cut, for instance, into two rolls 93 cm wide, one roll of width 108 cm, and a rest of 6 cm (which goes to waste).

Let us consider an order of

- 97 rolls of width 135 cm,
- 610 rolls of width 108 cm,
- 395 rolls of width 93 cm, and
- 211 rolls of width 42 cm.

What is the smallest number of 3 m rolls that have to be cut in order to satisfy this order, and how should they be cut?

In order to engage linear programming one has to be generous in introducing variables. We write down all of the requested widths: 135 cm, 108 cm, 93 cm, and 42 cm. Then we list all possibilities of cutting a 3 m paper roll into rolls of some of these widths (we need to consider only possibilities for which the wasted piece is shorter than 42 cm):

P1:	2×135	P7:	$108 + 93 + 2 \times 42$
P2:	$135 + 108 + 42$	P8:	$108 + 4 \times 42$
P3:	$135 + 93 + 42$	P9:	3×93
P4:	$135 + 3 \times 42$	P10:	$2 \times 93 + 2 \times 42$
P5:	$2 \times 108 + 2 \times 42$	P11:	$93 + 4 \times 42$
P6:	$108 + 2 \times 93$	P12:	7×42

For each possibility Pj on the list we introduce a variable $x_j \geq 0$ representing the number of rolls cut according to that possibility. We want to minimize the total number of rolls cut, i.e., $\sum_{j=1}^{12} x_j$, in such a way that the customers are satisfied. For example, to satisfy the demand for 395 rolls of width 93 cm we require

$$x_3 + 2x_6 + x_7 + 3x_9 + 2x_{10} + x_{11} \geq 395.$$

For each of the widths we obtain one constraint.

> For a more complicated order, the list of possibilities would most likely be produced by computer. We would be in a quite typical situation in which a linear program is not entered "by hand," but rather is generated by some computer program. More-advanced techniques even generate the possibilities "on the fly," during the solution of the linear program, which may save time and memory considerably. See the entry "column generation" in the glossary or Chvátal's book cited in Chapter 9, from which this example is taken.

The optimal solution of the resulting linear program has $x_1 = 48.5$, $x_5 = 206.25$, $x_6 = 197.5$, and all other components 0. In order to cut 48.5 rolls according to the possibility P1, one has to unwind half of a roll. Here we need more information about the technical possibilities of the paper mill: Is cutting a fraction of a roll technically and economically feasible? If yes, we have solved the problem optimally. If not, we have to work further and somehow take into account the restriction that only feasible solutions of the linear program with *integral* x_i are of interest. This is not at all easy in general, and it is the subject of Chapter 3.